CCGPS
Frameworks
Student Edition

Mathematics

CCGPS Analytic Geometry
Unit 5: Quadratic Functions

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"Making Education Work for All Georgians"

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Unit 5

Quadratic Functions

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OVERVIEW

In this unit students will:

- focuses on quadratic functions, equations, and applications
- explore variable rate of change
- learn to factor general quadratic expressions completely over the integers and to solve general quadratic equations by factoring by working with quadratic functions that model the behavior of objects that are thrown in the air and allowed to fall subject to the force of gravity
- learn to find the vertex of the graph of any polynomial function and to convert the formula for a quadratic function from standard to vertex form.
- explore quadratic inequalities graphically
- apply the vertex form of a quadratic function to find real solutions of quadratic equations that cannot be solved by factoring
- use exact solutions of quadratic equations to give exact values for the endpoints of the intervals in the solutions of quadratic inequalities.
- develop the concept of discriminant of a quadratic equation
- learn the quadratic formula
- explore complex numbers as non-real solutions of quadratic equations.
- explain why the graph of every quadratic function is a translation of the graph of the basic function $f(x) = x^2$
- justify the quadratic formula.

Although the units in this instructional framework emphasize key standards and big ideas at specific times of the year, routine topics such as estimation, mental computation, and basic computation facts should be addressed on an ongoing basis. Ideas related to the eight practice standards should be addressed constantly as well. To assure that this unit is taught with the appropriate emphasis, depth, and rigor, it is important that the tasks listed under “Evidence of Learning” be reviewed early in the planning process. A variety of resources should be utilized to supplement this unit. This unit provides much needed content information, but excellent learning activities as well. The tasks in this unit illustrate the types of learning activities that should be utilized from a variety of sources.

STANDARDS ADDRESSED IN THIS UNIT

KEY STANDARDS

Use complex numbers in polynomial identities and equations.

MCC9-12.N.CN.7 Solve quadratic equations with real coefficients that have complex solutions.
Interpret the structure of expressions

MCC9-12.A.SSE.1 Interpret expressions that represent a quantity in terms of its context.★

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(Focus on quadratic functions; compare with linear and exponential functions studied in Coordinate Algebra.)

MCC9-12.A.SSE.1a Interpret parts of an expression, such as terms, factors, and coefficients.★
(Focus on quadratic functions; compare with linear and exponential functions studied in Coordinate Algebra.)

MCC9-12.A.SSE.1b Interpret complicated expressions by viewing one or more of their parts as a single entity.★
(Focus on quadratic functions; compare with linear and exponential functions studied in Coordinate Algebra.)

MCC9-12.A.SSE.2 Use the structure of an expression to identify ways to rewrite it. (Focus on quadratic functions; compare with linear and exponential functions studied in Coordinate Algebra.)

Write expressions in equivalent forms to solve problems

MCC9-12.A.SSE.3 Choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.★
(Focus on quadratic functions; compare with linear and exponential functions studied in Coordinate Algebra.)

MCC9-12.A.SSE.3a Factor a quadratic expression to reveal the zeros of the function it defines.★

MCC9-12.A.SSE.3b Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines.★

Create equations that describe numbers or relationships
MCC9-12.A.CED.1 Create equations and inequalities in one variable and use them to solve problems. Include equations arising from linear and quadratic functions, and simple rational and exponential functions.★

MCC9-12.A.CED.2 Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.★
(Focus on quadratic functions; compare with linear and exponential functions studied in Coordinate Algebra.)

MCC9-12.A.CED.4 Rearrange formulas to highlight a quantity of interest, using the same reasoning as in solving equations.
(Focus on quadratic functions; compare with linear and exponential functions studied in Coordinate Algebra.)

Solve equations and inequalities in one variable

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MCC9-12.A.REI.4 Solve quadratic equations in one variable.

MCC9-12.A.REI.4a Use the method of completing the square to transform any quadratic equation in x into an equation of the form $(x – p)^2 = q$ that has the same solutions. Derive the quadratic formula from this form.

MCC9-12.A.REI.4b Solve quadratic equations by inspection
(e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as $a \pm bi$ for real numbers a and b.

Solve systems of equations

MCC9-12.A.REI.7 Solve a simple system consisting of a linear equation and a quadratic equation in two variables algebraically and graphically.
Interpret functions that arise in applications in terms of the context.

MCC9-12.F.IF.4 For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship. Key features include: intercepts; intervals where the function is increasing, decreasing, positive, or negative; relative maximums and minimums; symmetries; end behavior; and periodicity.★

MCC9-12.F.IF.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes.★
(Focus on quadratic functions; compare with linear and exponential functions studied in Coordinate Algebra.)

MCC9-12.F.IF.6 Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.★
(Focus on quadratic functions; compare with linear and exponential functions studied in Coordinate Algebra.)

Analyze functions using different representations

MCC9-12.F.IF.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.★
(Focus on quadratic functions; compare with linear and exponential functions studied in Coordinate Algebra.)

MCC9-12.F.IF.7a Graph linear and quadratic functions and show intercepts, maxima, and minima.★

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MCC9-12.F.IF.8 Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function. (Focus on quadratic functions; compare with linear and exponential functions studied in Coordinate Algebra.)

MCC9-12.F.IF.8a Use the process of factoring and completing the square in a quadratic function to show zeros, extreme values, and symmetry of the graph, and interpret these in terms of a context.

MCC9-12.F.IF.9 Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). (Focus on quadratic functions; compare with linear and exponential functions studied in Coordinate Algebra.)
Build a function that models a relationship between two quantities.

MCC9-12.F.BF.1 Write a function that describes a relationship between two quantities.★
(Focus on quadratic functions; compare with linear and exponential functions studied in Coordinate Algebra.)

MCC9-12.F.BF.1a Determine an explicit expression, a recursive process, or steps for calculation from a context. (Focus on quadratic functions; compare with linear and exponential functions studied in Coordinate Algebra.)

MCC9-12.F.BF.1b Combine standard function types using arithmetic operations. (Focus on quadratic functions; compare with linear and exponential functions studied in Coordinate Algebra.)

Build new functions from existing functions
MCC9-12.F.BF.3 Identify the effect on the graph of replacing f(x) by f(x) + k, k f(x), f(kx), and f(x + k) for specific values of k (both positive and negative); find the value of k given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology. Include recognizing even and odd functions from their graphs and algebraic expressions for them. (Focus on quadratic functions; compare with linear and exponential functions studied in Coordinate Algebra.)

Construct and compare linear, quadratic, and exponential models and solve problems.

MCC9-12.F.LE.3 Observe using graphs and tables that a quantity increasing exponentially eventually exceeds a quantity increasing linearly, quadratically, or (more generally) as a polynomial function.★

Summarize, represent, and interpret data on two categorical and quantitative variables

MCC9-12.S.ID.6 Represent data on two quantitative variables on a scatter plot, and describe how the variables are related.★

MCC9-12.S.ID.6a Fit a function to the data; use functions fitted to data to solve problems in the context of the data. Use given functions or choose a function suggested by the context. Emphasize linear, quadratic, and exponential models.

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RELATED STANDARDS

Standards for Mathematical Practice

The Standards for Mathematical Practice describe varieties of expertise that mathematics educators at all levels should seek to develop in their students. These practices rest on important "processes and proficiencies" with longstanding importance in mathematics education. The first of these are the NCTM process standards of problem solving, reasoning and proof, communication, representation, and connections. The second are the strands of mathematical proficiency specified in the National Research Council's report *Adding It Up*: adaptive reasoning, strategic competence, conceptual understanding (comprehension of mathematical concepts, operations and relations), procedural fluency (skill in carrying out procedures flexibly, accurately, efficiently and appropriately), and productive disposition (habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy).

1. **Make sense of problems and persevere in solving them.** High school students start to examine problems by explaining to themselves the meaning of a problem and looking for entry points to its solution. They analyze givens, constraints, relationships, and goals. They make conjectures about the form and meaning of the solution and plan a solution pathway rather than simply jumping into a solution attempt. They consider analogous problems, and try special cases and simpler forms of the original problem in order to gain insight into its solution. They monitor and evaluate their progress and change course if necessary. Older students might, depending on the context of the problem, transform algebraic expressions or change the viewing window on their graphing calculator to get the information they need. By high school, students can explain correspondences between equations, verbal descriptions, tables, and graphs or draw diagrams of important features and relationships, graph data, and search for regularity or trends. They check their answers to problems using different methods and continually ask themselves, "Does this make sense?" They can understand the approaches of others to solving complex problems and identify correspondences between different approaches.

2. **Reason abstractly and quantitatively.** High school students seek to make sense of quantities and their relationships in problem situations. They abstract a given situation and represent it symbolically, manipulate the representing symbols, and pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Students use quantitative reasoning to create coherent representations of the problem at hand; consider the units involved; attend to the meaning of quantities, not just how to compute them; and know and flexibly use different properties of operations and objects.

3. **Construct viable arguments and critique the reasoning of others.** High school students understand and use stated assumptions, definitions, and previously established results in constructing arguments. They make conjectures and build a logical progression of statements to explore the truth of their conjectures. They are able to analyze situations by breaking them into cases, and can recognize and use counterexamples. They justify their conclusions, communicate them to others, and respond to the arguments of others. They reason inductively

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about data, making plausible arguments that take into account the context from which the data arose. High school students are also able to compare the effectiveness of two plausible arguments, distinguish correct logic or reasoning from that which is flawed, and—if there is a flaw in an argument—explain what it is. High school students learn to determine domains to which an argument applies, listen or read the arguments of others, decide whether they make sense, and ask useful questions to clarify or improve the arguments.

4. **Model with mathematics.** High school students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. High school students making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

5. **Use appropriate tools strategically.** High school students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software. High school students should be sufficiently familiar with tools appropriate for their grade or course to make sound decisions about when each of these tools might be helpful, recognizing both the insight to be gained and their limitations. For example, high school students analyze graphs of functions and solutions generated using a graphing calculator. They detect possible errors by strategically using estimation and other mathematical knowledge. When making mathematical models, they know that technology can enable them to visualize the results of varying assumptions, explore consequences, and compare predictions with data. They are able to identify relevant external mathematical resources, such as digital content located on a website, and use them to pose or solve problems. They are able to use technological tools to explore and deepen their understanding of concepts.

6. **Attend to precision.** High school students try to communicate precisely to others by using clear definitions in discussion with others and in their own reasoning. They state the meaning of the symbols they choose, specifying units of measure, and labeling axes to clarify the correspondence with quantities in a problem. They calculate accurately and efficiently, express numerical answers with a degree of precision appropriate for the problem context. By the time they reach high school they have learned to examine claims and make explicit use of definitions.

7. **Look for and make use of structure.** By high school, students look closely to discern a pattern or structure. In the expression $x^2 + 9x + 14$, older students can see the 14 as 2×7 and the 9 as $2 + 7$. They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic

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expressions, as single objects or as being composed of several objects. For example, they can see $5 - 3(x - y)^2$ as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers x and y. High school students use these patterns to create equivalent expressions, factor and solve equations, and compose functions, and transform figures.

8. **Look for and express regularity in repeated reasoning.** High school students notice if calculations are repeated, and look both for general methods and for shortcuts. Noticing the regularity in the way terms cancel when expanding $(x - 1)(x + 1)$, $(x - 1)(x^2 + x + 1)$, and $(x - 1)(x^3 + x^2 + x + 1)$ might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, derive formulas or make generalizations, high school students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.

ENDURING UNDERSTANDINGS

- The graph of any quadratic function is a vertical and/or horizontal shift of a vertical stretch or shrink of the basic quadratic function $f(x) = x^2$.
- The vertex of a quadratic function provides the maximum or minimum output value of the function and the input at which it occurs.
- Every quadratic equation can be solved using the Quadratic Formula.
- The discriminant of a quadratic equation determines whether the equation has two real roots, one real root, or two complex conjugate roots.
- Quadratic equations can have complex solutions.

CONCEPTS/SKILLS TO MAINTAIN

It is expected that students will have prior knowledge/experience related to the concepts and skills identified below. It may be necessary to pre-assess in order to determine if time needs to be spent on conceptual activities that help students develop a deeper understanding of these ideas.

1. Use Function Notation
2. Put data into tables
3. Graph data from tables
4. Solve one variable linear equations
5. Determine domain of a problem situation
6. Solve for any variable in a multi-variable equation
7. Recognize slope of a linear function as a rate of change
8. Graph linear functions
9. Complex numbers
10. Graph inequalities

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SELECTED TERMS AND SYMBOLS

The following terms and symbols are often misunderstood. These concepts are not an inclusive list and should not be taught in isolation. However, due to evidence of frequent difficulty and misunderstanding associated with these concepts, instructors should pay particular attention to them and how their students are able to explain and apply them.

The definitions below are for teacher reference only and are not to be memorized by the students. Students should explore these concepts using models and real life examples. Students should understand the concepts involved and be able to recognize and/or demonstrate them with words, models, pictures, or numbers.

The website below is interactive and includes a math glossary.

http://intermath.coe.uga.edu/dictnary/homepg.asp
Definitions and activities for these and other terms can be found on the Intermath website. Links to external sites are particularly useful.

Formulas and Definitions:

- Horizontal shift: A rigid transformation of a graph in a horizontal direction, either left or right.

- Complete factorization over the integers: Writing a polynomial as a product of polynomials so that none of the factors is the number 1, there is at most one factor of degree zero, each polynomial factor has degree less than or equal to the degree of the product polynomial, each polynomial factor has all integer coefficients, and none of the factor polynomial can written as such a product.

- Vertex form of a quadratic function: A formula for a quadratic equation of the form
  - $f(x) = a(x - h)^2 + k$, where a is a nonzero constant and the vertex of the graph is the point (h, k).

- Discriminant of a quadratic equation: The discriminant of a quadratic equation of the form
  - $ax^2 + bx + c = 0$, $a \neq 0$, is the number $b^2 - 4ac$.

Theorems:

For $h = \dfrac{-b}{2a}$ and $k = f\left(\dfrac{-b}{2a}\right)$, $f(x) = a(x - h)^2 + k$ is the same function as $f(x) = ax^2 + bx + c$.

The graph of any quadratic function can be obtained from transformations of the graph of the basic function $f(x) = x^2$.

Quadratic formula: The solution(s) of the quadratic equation of the form $ax^2 + bx + c = 0$, where a, b, and c are real numbers with $a \neq 0$, is

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$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The discriminant of a quadratic equation is positive, zero, or negative if and only if the equation has two real solutions, one real solution, or two complex conjugate number solutions respectively.

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Just Jogging

For distances of 12 miles or less, a certain jogger can maintain an average speed of 6 miles per hour while running on level ground. (Distance = rate*time)

1. If this jogger runs around a level track at an average speed of 6 mph, how long in hours will the jogger take to run each of the following distances? [Express your answers as fractions of an hour in simplest form.]

   (a) 3 miles (b) 9 miles (c) 1 mile (d) ½ mile (e) 1/10 mile

2. Write an algebraic expression for the time (t) it takes in hours for this jogger to run (*x)* miles on level ground at an average speed of 6 miles per hour.
3. Each day this jogger warms up with stretching exercises for 15 minutes, jogs for a while, and then cools down for 15 minutes. How long would this exercise routine take, in hours, if the jogger ran for 5 miles? [Express your answer as a fraction in simplest form.]

4. Let *T* represent the total time in hours it takes for this workout routine when the jogger runs for *x* miles. Write a formula for calculating *T* given *x*, where, as in Question 2, *x* is number of miles the jogger runs. Express the formula for *T* as a single algebraic fraction.

5. If the jogger skipped the warm-up and cool-down period and used this additional time to jog, how many more miles would be covered? Does this answer have any connection to the answer to question 4 above?

Suppose this same jogger decides to go to a local park and use one of the paths there for the workout routine one day each week. This path is a gently sloping one that winds its way to the top of a hill.

6. If the jogger can run at an average speed of 5.5 miles per hour up the slope and 6.5 miles per hour going down the slope, how long, in hours, will it take for the jogger to cover 2 miles by going uphill for 1 mile and then returning 1 mile back down the hill? Give an exact answer expressed as a fraction in simplest terms and then give a decimal approximation correct to three decimal places.
7. If the jogger can run at an average speed of 5.3 miles per hour up the slope and 6.7 miles per hour going down the slope, how long, in hours, will it take for the jogger to cover 2 miles by going uphill for 1 mile and then returning 1 mile back down the hill? Give an exact answer expressed as a fraction in simplest terms and then give a decimal approximation correct to three decimal places.
8. Write an algebraic expression for the total time, in hours, that it takes the jogger to cover 2 miles by going uphill for 1 mile and then returning 1 mile back down the hill if the jogger runs uphill at an average speed that is *c* miles per hour slower than the level-ground speed of 6 miles per hour and runs downhill at an average speed that is *c* miles per hour faster than the level-ground speed of 6 miles per hour. Simplify your answer to a single algebraic fraction. Verify that your expression gives the correct answers for Questions 6 and 7.

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9. The average speed in miles per hour is defined to be the distance in miles divided by the time in hours spent covering the distance.
   (a) What is the jogger's average speed for a two mile trip on level ground?

   (b) What is the jogger's average speed for the two mile trip in question 6?

   (c) What is the jogger's average speed for the two mile trip in question 7?

   (d) Write an expression for the jogger's average speed over the two-mile trip (one mile up and one mile down) when the average speed uphill is c miles per hour slower than the level-ground speed of 6 miles per hour and the average speed downhill at an average speed that is c miles per hour faster than the level-ground speed of 6 miles per hour. Express your answer as a simplified algebraic fraction.

   (e) Use the expression in part (d) to recalculate your answers for parts (b) and (c)? What value of c should you use in each part?

10. For what value of c would the jogger's average speed for the two-mile trip (one mile up and one mile down) be 4.5 miles per hour? For this value of c, what would be the jogger's average rate uphill and downhill?

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Paula's Peaches

Paula is a peach grower in central Georgia and wants to expand her peach orchard. In her current orchard, there are 30 trees per acre and the average yield per tree is 600 peaches. Data from the local agricultural experiment station indicates that if Paula plants more than 30 trees per acre, once the trees are in production, the average yield of 600 peaches per tree will decrease by 12 peaches for each tree over 30. She needs to decide how many trees to plant in the new section of the orchard.

1. Paula believes that algebra can help her determine the best plan for the new section of orchard and begins by developing a mathematical model of the relationship between the number of trees per acre and the average yield in **peaches per tree.**
   a. Is this relationship linear or nonlinear? Explain your reasoning.

   b. If Paula plants 6 more trees per acre, what will be the **average** yield in peaches per tree? What is the **average** yield in peaches per tree if she plants 42 trees per acre?

   c. Let T be the function for which the input x is the number of trees planted on each acre and $T(x)$ is the average yield in peaches per tree. Write a formula for $T(x)$ in terms of x and express it in simplest form. Explain how you know that your formula is correct.

   d. Draw a graph of the function T. Given that the information from the agricultural experiment station applies only to increasing the number of trees per acre, what is an appropriate domain for the function T?

2. Since her income from peaches depends on the total number of peaches she produces, Paula realized that she needed to take a next step and consider the total number of peaches that she can produce **per acre**.

   a. With the current 30 trees per acre, what is the yield in total peaches per acre? If Paula plants 36 trees per acre, what will be the yield in total peaches per acre? 42 trees per acre?

   b. Find the average rate of change of peaches per acre with respect to number of trees per acre when the number of trees per acre increases from 30 to 36. Write a sentence to explain what this number means.

   c. Find the average rate of change of peaches per acre with respect to the number of trees per acre when the number of trees per acre increases from 36 to 42. Write a sentence to explain the meaning of this number.

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d. Is the relationship between number of trees per acre and yield in peaches per acre linear? Explain your reasoning.

e. Let Y be the function that expresses this relationship; that is, the function for which the input x is the number of trees planted on each acre and the output $Y(x)$ is the total yield in peaches per acre. Write a formula for $Y(x)$ in terms of x and express your answer in expanded form.

f. Calculate $Y(30)$, $Y(36)$, and $Y(42)$. What is the meaning of these values? How are they related to your answers to parts a through c?

g. What is the relationship between the domain for the function T and the domain for the function Y? Explain.

3. Paula wants to know whether there is a different number of trees per acre that will give the same yield per acre as the yield when she plants 30 trees per acre.

a. Write an equation that expresses the requirement that x trees per acre yields the same total number of peaches per acre as planting 30 trees per acre.

b. Use the algebraic rules for creating equivalent equations to obtain an equivalent equation with an expression in x on one side of the equation and 0 on the other.

c. Multiply this equation by an appropriate rational number so that the new equation is of the form $x^2 + bx + c = 0$. Explain why this new equation has the same solution set as the equations from parts a and b.

d. When the equation is in the form $x^2 + bx + c = 0$, what are the values of b and c

e. Find integers m and n such that $m \cdot n = c$ and $m + n = b$.

f. Using the values of m and n found in part e, form the algebraic expression $(x+m)(x+n)$ and simplify it.

g. Combining parts d through f, rewrite the equation from part c in the form $(x+m)(x+n) = 0$.

h. This equation expresses the idea that the product of two numbers, $x+m$ and $x+n$, is equal to 0. We know from the discussion in Unit 2 that, when the product of two numbers is 0, one of the numbers has to be 0. This property is called the ***Zero Factor Property***. For these particular values of m and n, what value of x makes $x+m=0$ and what value of x makes $x+n=0$?

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i. Verify that the answers to part h are solutions to the equation written in part a. It is appropriate to use a calculator for the arithmetic.

j. Write a sentence to explain the meaning of your solutions to the equation in relation to planting peach trees.

4. Paula saw another peach grower, Sam, from a neighboring county at a farm equipment auction and began talking to him about the possibilities for the new section of her orchard. Sam was surprised to learn about the agricultural research and said that it probably explained the drop in yield for a orchard near him. This peach farm has more than 30 trees per acre and is getting an average total yield of 14,400 peaches per acre. *(Remember: Throughout this task assume that, for all peach growers in this area, the average yield is 600 peaches per tree when 30 trees per acre are planted and that this yield will decrease by 12 peaches per tree for each additional tree per acre.)*

a. Write an equation that expresses the situation that x trees per acre results in a total yield per acre of 14,400 peaches per acre.

b. Use the algebraic rules for creating equivalent equations to obtain an equivalent equation with an expression in x on one side of the equation and 0 on the other.

c. Multiply this equation by an appropriate rational number so that the new equation is of the form $x^2 + bx + c = 0$. Explain why this new equation has the same solution set as the equations from parts and b.

d. When the equation is in the form $x^2 + bx + c = 0$, what is value of b and what is the value of c?

e. Find integers m and n such that $m \cdot n = c$ and $m + n = b$.

f. Using the values of m and n found in part e, form the algebraic expression $(x+m)(x+n)$ and simplify it.

g. Combining parts d through f, rewrite the equation from part d in the form $(x+m)(x+n)=0$.

h. This equation expresses the idea that the product of two numbers, $x+m$ and $x+n$, is equal to 0. We know from the discussion in Unit 2 that, when the product of two numbers is 0, one of the numbers has to be 0. What value of x makes $x+m=0$? What value of x makes $x+n=0$?

i. Verify that the answers to part h are solutions to the equation written in part a. It is appropriate to use a calculator for the arithmetic.

j. Which of the solutions verified in part i is (are) in the domain of the function Y? How many peach trees per acre are planted at the peach orchard getting 14400 peaches per acre?

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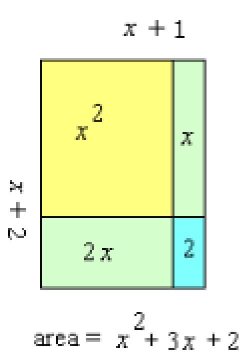
The steps in items 3 and 4 outline a method of solving equations of the form $x^2 + bx + c = 0$. These equations are called ***quadratic equations*** and an expression of the form $x^2 + bx + c$ is called a ***quadratic expression***. In general, quadratic expressions may have any nonzero coefficient on the x^2 term. An important part of this method for solving quadratic equations is the process of rewriting an expression of the form $x^2 + bx + c$ in the form $(x+m)(x+n)$. The identity tells us that the product of the numbers *m* and *n* must equal *c* and that the sum of *m* and *n* must equal *b*.

5. Since the whole expression $(x+m)(x+n)$ is a product, we call the expressions $x+m$ and $x+n$ the **factors** of this product. For the following expressions in the form $x^2 + bx + c$, rewrite the expression as a product of factors of the form *x* + *m* and *x* + *n*. Verify each answer by drawing a rectangle with sides of length $x+m$ and $x+n$, respectively, and showing geometrically that the area of the rectangle is $x^2 + bx + c$.

Example: $x^2 + 3x + 2$
Solution: (x + 1)*(x + 2)



a. $x^2 + 6x + 5$

b. $x^2 + 5x + 6$

c. $x^2 + 7x + 12$

On a separate sheet of paper:

d. $x^2 + 8x + 12$

e. $x^2 + 13x + 36$

f. $x^2 + 13x + 12$

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6. In item 5, the values of *b* and *c* were positive. Now use Identity 1 in reverse to factor each of the following quadratic expressions of the form $x^2 + bx + c$ where *c* is positive but *b* is negative. Verify each answer by multiplying the factored form to obtain the original expression.

On a separate sheet of paper:

a. $x^2 - 8x + 7$

b. $x^2 - 9x + 18$

c. $x^2 - 4x + 4$

d. $x^2 - 8x + 15$

e. $x^2 - 11x + 24$

f. $x^2 - 11x + 18$

g. $x^2 - 12x + 27$

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Paula's Peaches Continued!

7. Now we return to the peach growers in central Georgia. How many peach trees per acre would result in only 8400 peaches per acre? Which answer makes sense in this particular context?

8. If there are no peach trees on a property, then the yield is zero peaches per acre. Write an equation to express the idea that the yield is zero peaches per acre with x trees planted per acre, where x is number greater than 30. Is there a solution to this equation? Explain why only one of the solutions makes sense in this context.

9. At the same auction where Paula heard about the peach grower who was getting a low yield, she talked to the owner of a major farm supply store in the area. Paula began telling the store owner about her plans to expand her orchard, and the store owner responded by telling her about a local grower that gets 19,200 peaches per acre. Is this number of peaches per acre possible? If so, how many trees were planted?

10. Using graph paper, explore the graph of Y as a function of x.
    a. What points on the graph correspond to the answers for part j from questions 3 and 4?

    b. What points on the graph correspond to the answers to questions 7, 8, and 9?

    c. What is the relationship of the graph of the function Y to the graph of the function f , where the formula for $f(x)$ is the same as the formula for $Y(x)$ but the domain for f is all real numbers?

    d. Questions 4, 7, and 8 give information about points that are on the graph of f but not on the graph of Y. What points are these?

    e. Graph the functions f and Y on the same axes. How does your graph show that the domain of f is all real numbers? How is the domain of Y shown on your graph?

    f. Draw the line $y = 18000$ on the graph drawn for item 10, part e. This line is the graph of the function with constant value 18000. Where does this line intersect the graph of the function Y? Based on the graph, how many trees per acre give a yield of more than 18000 peaches per acre?

    g. Draw the line $y = 8400$ on your graph. Where does this line intersect the graph of the function Y? Based on the graph, how many trees per acre give a yield of <u>fewer than</u> 8400 peaches per acre?

    h. Use a graphing utility and this intersection method to find the number of trees per acre that give a total yield **closest** to the following numbers of peaches per acre:
    (i) 10000 (ii) 15000 (iii) 20000

    i. Find the value of the function Y for the number of trees given in answering (i) – (iii) in part c above.

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11. In items 5 and 6, we used factoring as part of a process to solve equations that are equivalent to equations of the form $x^2 + bx + c = 0$ where b and c are integers. Look back at the steps you did in items 3 and 4, and describe the process for solving an equation of the form $x^2 + bx + c = 0$. Use this process to solve each of the following equations, that is, to find all of the numbers that satisfy the original equation. Verify your work by checking each solution in the original equation.

    a. $x^2 - 6x + 8 = 0$

    b. $x^2 - 15x + 36 = 0$

    c. $x^2 + 28x + 27 = 0$

    d. $x^2 - 3x - 10 = 0$

    e. $x^2 + 2x - 15 = 0$

    f. $x^2 - 4x - 21 = 0$

    g. $x^2 - 7x = 0$

    h. $x^2 + 13x = 0$

12. For each of the equations solved in question 11, do the following.
    a. Use technology to graph a function whose formula is given by the left-hand side of the equation.
    b. Find the points on the graph which correspond to the solutions found in question 8.
    c. How is each of these results an example of the intersection method explored above?

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PROTEIN BAR TOSS

Blake and Zoe were hiking in a wilderness area. They came up to a scenic view at the edge of a cliff. As they stood enjoying the view, Zoe asked Blake if he still had some protein bars left, and, if so, could she have one. Blake said, "Here's one; catch!" As he said this, he pulled a protein bar out of his backpack and threw it up to toss it to Zoe. But the bar slipped out of his hand sooner than he intended, and the bar went straight up in the air with his arm out over the edge of the cliff. The protein bar left Blake's hand moving straight up at a speed of 24 feet per second. If we let *t* represent the number of seconds since the protein bar left the Blake's hand and let *h*(*t*) denote the height of the bar, in feet above the ground at the base of the cliff, then, assuming that we can ignore the air resistance, we have the following formula expressing *h*(*t*) as a function of *t*,

$$h(t) = -16t^2 + 24t + 160.$$

1. **Use technology to graph the equation** $y = -16t^2 + 24t + 160$. Remember t represents time and y represents height. Find a viewing window that includes the part of this graph that corresponds to the situation with Blake and his toss of the protein bar. What viewing window did you select? **Sketch** the graph below.

2. What was the height of the protein bar, measured from the ground at the base of the cliff, at the instant that it left Blake's hand? What special point on the graph is associated with this information?

3. At what other time does the protein bar reach the height from question 2? Describe how you reached your answer.

4. If Blake does not catch the falling protein bar, how long will it take for the protein bar to hit the base of the cliff? Justify your answer graphically. Then write a quadratic equation that you would need to solve to justify the answer algebraically (*Note: you do not need to solve this equation at this point*).

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The equation from item 5 can be solved by factoring, but it requires factoring a quadratic polynomial where the leading coefficient is not 1. Our next goal is to learn about factoring this type of polynomial. We start by examining products that lead to such quadratic polynomials.

5. For each of the following, perform the indicated multiplication and use a *rectangular model* to show a geometric interpretation of the product as area for positive values of *x*.

a. $(2x+3)(3x+4)$ b. $(x+2)(4x+11)$ c. $(2x+1)(5x+4)$

6. For each of the following, perform the indicated multiplication (*Note: you do not need to use the rectangular model)*.

a. $(2x-3)(9x+2)$ b. $(3x-1)(x-4)$ c. $(4x-7)(2x+9)$

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Factoring Polynomials

The method for factoring general quadratic polynomial of the form $ax^2 + bx + c$, with *a, b,* and *c* all non-zero integers, is similar to the method previously learned for factoring quadratics of this form but with the value of *a* restricted to *a* = 1. The next item guides you through an example of this method.

7. **Factor** the quadratic polynomial $6x^2 + 7x - 20$ using the following steps.

   a. Think of the polynomial as fitting the form $ax^2 + bx + c$.

   What is *a*? ____ What is *c*? ____ What is the product *ac*? ____

   b. List all possible pairs of integers such that their product is equal to the number *ac*. It may be helpful to organize your list in a table. Make sure that your integers are chosen so that their product has the same sign, positive or negative, as the number *ac* from above, and make sure that you list all of the possibilities.

Integer pair	*Integer pair*	*Integer pair*	*Integer pair*

   c. What is *b* in the quadratic polynomial given? _____ Add the integers from each pair listed in part b. Which pair adds to the value of *b* from your quadratic polynomial? We'll refer to the integers from this pair as *m* and *n*.

Integer pair, sum		*Integer pair, sum*		*Integer pair, sum*		*Integer pair, sum*	

   d. Rewrite the polynomial replacing *bx* with $mx + nx$. [*Note either m or n could be negative; the expression indicates to add the terms mx and nx including the correct sign.*]

   e. Factor the polynomial from part d by grouping.

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f. Check your answer by performing the indicated multiplication in your factored polynomial. Did you get the original polynomial back?

8. Use the method outlined in the steps of item 8 to factor each of the following quadratic polynomials. Is it necessary to always list all of the integer pairs whose product is *ac*? Explain your answer.

a. $2x^2 + 3x - 54$

b. $4w^2 - 11w + 6$

c. $3t^2 - 13t - 10$

d. $8x^2 + 5x - 3$

e. $18z^2 + 17z + 4$

f. $6p^2 - 49p + 8$

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9. If a quadratic polynomial can be factored in the form $(Ax+B)(Cx+D)$, where the *A*, *B*, *C*, and *D* are all integers, the method you have been using will lead to the answer, specifically called the correct ***factorization***. Show that the following quadratic polynomial <u>cannot</u> be factored in the form $(Ax+B)(Cx+D)$, where the *A, B, C,* and *D* are all integers. Factor the following: $4z^2+z-6$

Integer pair, sum		Integer pair, sum		Integer pair, sum		Integer pair, sum	

10. The method required to solve the factored quadratic polynomial is the *Zero Factor Property*. Use your factorizations from item 8 as you solve the quadratic equations below (*Note: You factored these already, just make sure each equation is set equal to zero before you begin*).

a. $2x^2+3x-54=0$

b. $4w^2+6=11w$

c. $3t^2-13t=10$

d. $2x(4x+3)=3+x$

e. $18z^2+21z=4(z-1)$

f. $8-13p=6p(6-p)$

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11. Now we return to our goal of solving the equation from item 4. Solve the quadratic equation using factorization and the Zero Factor Property. Explain how the solution relates to the real-world situation of throwing a protein bar. Do both of your solutions make sense?

12. Suppose the cliff had been 56 feet higher. Answer the following questions for this higher cliff.
    a. What was the height of the protein bar, measured from the ground at the base of the cliff, at the instant that it left Blake's hand? What special point on the graph is associated with this information?

    b. What is the formula for the height function in this situation?

    c. If Blake wants to catch the falling protein bar, how long does he have until it hits the ground below the cliff? Justify your answer algebraically.

    d. If Blake does not catch the falling protein bar, how long it take for the protein bar to hit the ground below the cliff? Justify your answer algebraically.

13. Suppose the cliff had been 88 feet lower. Answer the following questions for this lower cliff.

    a. What was the height of the protein bar, measured from the ground at the base of the cliff, at the instant that it left Blake's hand? What special point on the graph is associated with this information?

    b. What is the formula for the height function in this situation?

    c. If Blake does not catch the falling protein bar, how long it take for the protein bar to hit the ground below the cliff? Justify your answer algebraically.

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The Protein Bar Toss, Part 2

In the first part of the learning task about Blake attempting to toss a protein bar to Zoe, you found how long it took for the bar to go up and come back down to the starting height. However, there is a question we did not consider: How high above its starting point did the protein bar go before it started falling back down? We're going to explore that question now.

1. So far in Unit 5, you have examined the graphs of many different quadratic functions. Consider the functions you graphed in the first part of the Protein Bar Toss. Each of these functions has a formula that is, or can be put in, the form $y = ax^2 + bx + c$ with $a \neq 0$. When we consider such formulas with domain all real numbers, there are some similarities in the shapes of the graphs. The shape of each graph is called a *parabola*. List at least three characteristics common to the parabolas seen in these graphs.

2. The question of how high the protein bar goes before it starts to come back down is related to a special point on the graph of the function. This point is called the *vertex* of the parabola. What is special about this point?

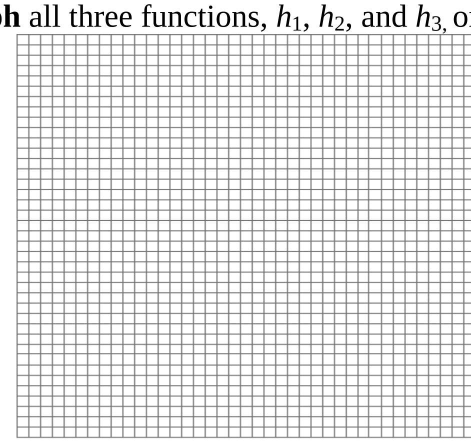
3. In the first part of the protein bar task you considered three different functions, each one corresponding to a different cliff height. Let's rename the first of these functions as h_1, so that

$$h_1(t) = -16t^2 + 24t + 160.$$

   a. Let $h_2(t)$ denote the height of the protein bar if it is thrown from a cliff that is 56 feet higher. Write the formula for the function h_2.

   b. Let $h_3(t)$ denote the height of the protein bar if it is thrown from a cliff that is 88 feet lower. Write the formula for the function h_3.

   c. **Use technology to graph** all three functions, h_1, h_2, and h_3, on the same axes.



   d. Estimate the coordinates of the vertex for each graph.

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e. What number do the coordinates have in common? What is the meaning of this number in relation to the toss of the protein bar?

f. The other coordinate is different for each vertex. Explain the meaning of this number for each of the vertices.

4. Consider the formulas for h_1, h_2, and h_3.
   a. How are the formulas different?

   b. Based on your answer to part a, how are the three graphs related? Do you see this relationship in your graphs of the three functions on the same axes?

Estimating the vertex from the graph gives us an approximate answer to our original question, but an algebraic method for finding the vertex would give us an exact answer.

5. For each of the quadratic functions below, find the *y*-intercept of the graph **<u>and</u>** all other points with this value for the *y*-coordinate.

   a. $f(x) = x^2 - 4x + 9$

   b. $f(x) = 4x^2 + 8x - 5$

   c. $f(x) = -x^2 - 6x + 7$

   d. $f(x) = ax^2 + bx + c,\ \ a \neq 0$

6. One of the characteristics of a parabola graph is that the graph has a line symmetry.
   a. For each of the parabolas considered in item 5, use what you know about the graphs of quadratic functions in general with the specific information you have about these particular functions to find an equation for the <u>line of symmetry</u>.

   b. The line of symmetry for a parabola is called the *axis of symmetry*. Explain the relationship between the axis of symmetry and the vertex of a parabola.

   c. Find the *y*-coordinate of the vertex for the quadratic functions in item 5, parts a, b, and c, and then state the vertex as a point.

   d. Describe a method for finding the vertex of the graph of any quadratic function given in the form $f(x) = ax^2 + bx + c,\ \ a \neq 0$.

7. Return to height functions $h_1(t)=-16t^2+24t+160$, $h_2(t)=-16t^2+24t+216$, and $h_3(t)=-16t^2+24t+72$.
   a. Use the method you described in item 6, part d, to find the exact coordinates of the vertex of each

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b. Find the exact answer to the question: How high above its starting point did the protein bar go before it started falling back down?

8. Each part below gives a list of functions. Describe the geometric transformation of the graph of first function that results in the graph of the second, and then describe the transformation of the graph of the second that gives the graph of the third, and, where applicable, describe the transformation of the graph of the third that yields the graph of the last function in the list.

   a. $f(x) = x^2,\ \ f(x) = x^2 + 5,\ \ f(x) = (x-2)^2 + 5$

   b. $f(x) = x^2,\ \ f(x) = 4x^2,\ \ f(x) = 4x^2 - 9,\ \ f(x) = 4(x+1)^2 - 9,$

   c. $f(x) = x^2,\ \ f(x) = -x^2,\ \ f(x) = -x^2 + 16,\ \ f(x) = -(x+3)^2 + 16$

9. Expand each of the following formulas from the vertex form $f(x) = a(x-h)^2 + k$ to the standard form $\boldsymbol{f(x) = ax^2 + bx + c}$.

   a. f(x)=(x - 2)^2+5

   b. f(x)=4(x + 1)^2 - 9

   c. f(x)= - (x + 3)^2+ 16

   d. Compare these expanded formulas to #5. How are they related?

   e. Compare the vertices in the original and expanded form. What special property do you notice?

10. For any quadratic function of the form $f(x) = a(x-h)^2 + k$:

    a. What do the *h* and *k* in the formula represent relative to the function?

    b. Is there an alternative way to find (h, k) without finding two symmetrical points, finding the midpoint, and then finding the corresponding y value to the midpoint?

11. Use the **vertex form** of the equations for the functions h_1, h_2, and h_3 you found in #7 to verify algebraically the equivalence with the original formulas for the functions.

12. For the functions given below, *put the formula in the vertex* form $f(x) = a(x-h)^2 + k$**,** give the *equation of the axis of symmetry*, and *describe how to transform* the graph of $y = x^2$ to create the graph of the given function.

    a. $f(x) = 3x^2 + 12x + 13$

    b. $f(x) = x^2 - 7x + 10$

    c. $f(x) = -2x^2 + 12x - 24$

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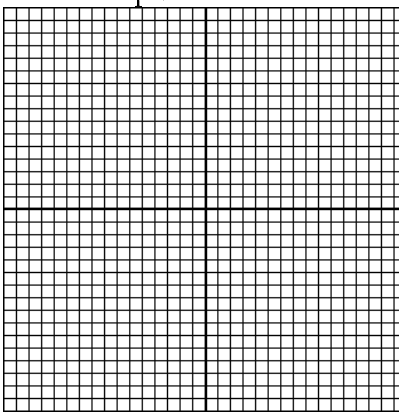
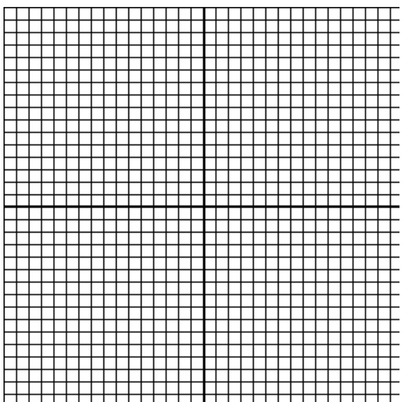
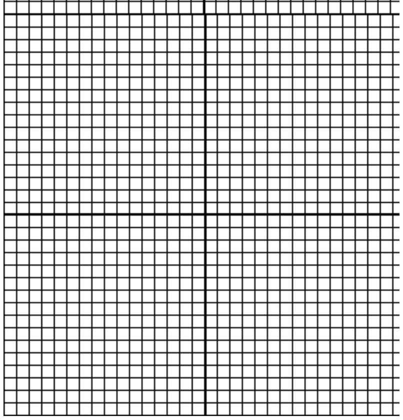
13. Make a hand-drawn sketch of the graphs of the functions in item 12. Make a dashed line for the axis of symmetry, and plot the vertex, *y*-intercept and the point symmetric with the *y*-intercept.







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14. Which of the graphs that you drew in item 13 have *x*-intercepts?

    a. Find the *x*-intercepts that do exist by solving an appropriate equation and then add the corresponding points to your sketch(es) from item 13.

    b. Explain geometrically why some of the graphs have *x*-intercepts and some do not.

    c. Explain how to use the vertex form of a quadratic function to decide whether the graph of the function will or will not have *x*-intercepts. Explain your reasoning.

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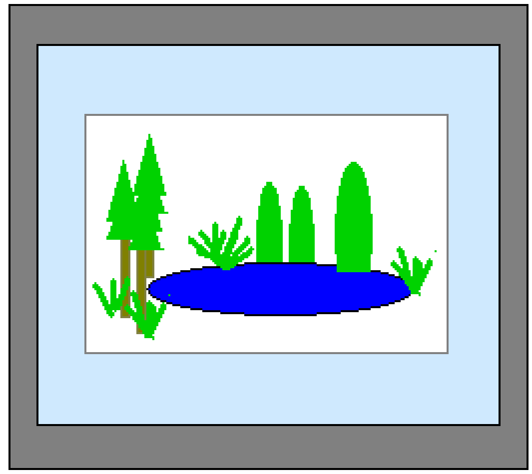

1. Hannah is an aspiring artist who enjoys taking nature photographs with her digital camera. Her mother, Cheryl, frequently attends estate sales in search of unique decorative items. Last month Cheryl purchased an antique picture frame that she thinks would be perfect for framing one of Hannah's recent photographs. The frame is rather large, so the photo needs to be enlarged. Cheryl wants to mat the picture. One of Hannah's art books suggest that mats should be designed so that the picture takes up 50% of the area inside the frame and the mat covers the other 50%. The inside of the picture frame is 20 inches by 32 inches. Cheryl wants Hannah to enlarge, and crop if necessary, her photograph so that it can be matted with a mat of uniform width and, as recommended, take up 50% of the area inside the mat. See the image at the right.

a. Let x denote the width of the mat for the picture. Model this situation with a diagram. Write an equation in x that models this situation.

b. Put the equation from part a in the standard form $ax^2 + bx + c = 0$. Can this equation be solved by factoring (using integers)?

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c. **<u>The quadratic formula</u>** can be used to solve quadratic equations that cannot be solved by factoring over the integers. Using the simplest equivalent equation in standard form, identify *a*, *b*, and *c* from the equation in part b and find $b^2 - 4ac$; then substitute these values in the quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, to find the solutions for *x*. Give exact answers for *x* and approximate the solutions to two decimal places.

d. To the nearest tenth of an inch, what should be the width of the mat and the dimensions for the photo enlargement?

2. The quadratic formula can be very useful in solving problems. Thus, it should be practiced enough to develop accuracy in using it and to allow you to commit the formula to memory. Use the quadratic formula to solve each of the following quadratic equations, even if you could solve the equation by other means. Begin by identifying *a*, *b*, and *c* and finding $b^2 - 4ac$; then substitute these values into the formula.

a. $4z^2 + z - 6 = 0$

b. $t^2 + 2t + 8 = 0$

c. $3x^2 + 15x = 12$

d. $25w^2 + 9 = 30w$

e. $7x^2 = 10x$

f. $\dfrac{t}{2} + \dfrac{7}{t} = 2$

g. $3\left(2p^2 + 5\right) = 23p$

h. $12z^2 = 90$

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3. The expression $b^2 - 4ac$ in the quadratic formula is called the **discriminant** of the quadratic equation in standard form. All of the equations in item 2 had values of a, b, and c that are rational numbers. Answer the following questions for quadratic equations in standard form when **a, *b*, and *c* are rational numbers**. Make sure that your answers are consistent with the solutions from item 2.
   a. What kind of number is the discriminant when there are two real number solutions to a quadratic equation?

   b. What kind of number is the discriminant when the two real number solutions to a quadratic equation are rational numbers?

   c. What kind of number is the discriminant when the two real number solutions to a quadratic equation are irrational numbers?

   d. Could a quadratic equation with rational coefficients have one rational solution and one irrational solution? Explain your reasoning.

   e. What kind of number is the discriminant when there is only one real number solution? What kind of number do you get for the solution?

   f. What kind of number is the discriminant when there is no real number solution to the equation?

4. There are many ways to show why the quadratic formula always gives the solution(s) to any quadratic equation with real number coefficients. You can work through one of these by responding to the parts below. Start by assuming that each of a, b, and c is a real number, that $a \neq 0$, and then consider the quadratic equation $ax^2 + bx + c = 0$.
   a. Why do we assume that $a \neq 0$?

   b. Form the corresponding quadratic function, $f(x) = ax^2 + bx + c$, and put the formula for f(x) in vertex form, expressing k in the vertex form as a single rational expression.

   c. Use the vertex form to solve for x-intercepts of the graph and simplify the solution. Hint: Consider two cases, $a > 0$ and $a < 0$, in simplifying $\sqrt{a^2}$.

5. Use the quadratic formula to solve the following equations with real number coefficients. Approximate each real, but irrational, solution correct to hundredths.
   a. $x^2 + \sqrt{5}x + 1 = 0$
   b. $3q^2 - 5q + 2\pi = 0$
   c. $3t^2 + 11 = 2\sqrt{33}t$
   d. $9w^2 = \sqrt{13}w$

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6. Verify each answer for item 5 by using a graphing utility to find the *x*-intercept(s) of an appropriate quadratic function.

   a. Put the function for item 5, part c, in vertex form. Use the vertex form to find the x-intercept.

   b. Solve the equation from item 5, part d, by factoring.

7. Answer the following questions for quadratic equations in standard from where ***a*, *b*, and *c* are real numbers**.
   a. What kind of number is the discriminant when there are two real number solutions to a quadratic equation?

   b. Could a quadratic equation with real coefficients have one rational solution and one irrational solution? Explain your reasoning.

   c. What kind of number is the discriminant when there is only one real number solution?

   d. What kind of number is the discriminant when there is no real number solution to the equation?

   e. Summarize what you know about the relationship between the determinant and the solutions of a quadratic of the form $ax^2 + bx + c = 0$ where *a*, *b*, and *c* are real numbers with $a \neq 0$ into a formal statement using biconditionals.

8. A landscape designer included a cloister (a rectangular garden surrounded by a covered walkway on all four sides) in his plans for a new public park. The garden is to be 35 feet by 23 feet and the total area enclosed by the garden and walkway together is 1200 square feet. To the nearest inch, how wide does the walkway need to be?

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9. In another area of the park, there will be an open rectangular area of grass surrounded by a flagstone path. If a person cuts across the grass to get from the southeast corner of the area to the northwest corner, the person will walk a distance of 15 yards. If the length of the area is 5 yards more than the width, to the nearest foot, how much distance is saved by cutting across rather than walking along the flagstone path?

10. What if you are solving a quadratic equation by using the quadratic formula, and you get a discriminant that is a negative number? Recalling your work in a previous unit, a negative square root can be written as a complex number using *i*. Solve each of the following quadratic equations using the quadratic formula and writing the answers as complex numbers.

a) $x^2 + x = -6$ b) $-2x^2 + 12x - 24 = 0$ c) $5x^2 - 2x = -29$

d) $-3x^2 + x - 33 = 0$ e) $x^2 + 2x = -15$

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Parabola Investigation

For this investigation I have not provided a worksheet because I want you to gain some practice in efficiently keeping and effectively organizing your notes in order to recognize patterns and describe them. As you use the graphing calculator to investigate each question be sure to keep track of all the equations you try along with their resulting graphs. What doesn't work in one case may be the key to another question that will come later.

a) Graph the parabola $y = x^2$. Make an accurate sketch of the graph. Be sure to label any important points on your graph. In addition to x– and y–intercepts be sure to label the lowest point which is called the vertex.

b) Find a way to change the equation to make the same parabola *open downward*. The new parabola should be congruent (the same shape and size) to $y = x^2$, with the same vertex, except it should open downward so its vertex will be its highest point. Record the equations you tried, along with their results. Write down the results even when they are wrong, they may come in handy below.

c) Find a way to change the equation to make the $y = x^2$ parabola *stretch vertically*(it will appear steeper). The new parabola should have the same vertex and orientation (i.e., open up) as $y = x^2$. Record the equations you tried, along with their results and your observations.

d) Find a way to change the equation to make the $y = x^2$ parabola *compress vertically*(it will appear as if the points in $y = x^2$move toward the x–axis). Record the equations you try, their results, and your observations.

e) Find a way to change the equation to make the $y = x^2$ parabola *move 5 units down*. That is, your new parabola should look exactly like $y = x^2$, but the vertex should be at (0, –5) Record the equations you try, along with their results. Include a comment about moving the graph up as well as down. Record the equations you try, along with their results. Include a comment about moving the graph up as well as down.

f) Find a way to change the equation to make the $y = x^2$ parabola *move 3 units to the right*. That is, your new parabola should look exactly like $y = x^2$, except that the vertex should be at the point (3, 0). Record the equations you try, along with their results. Tell how to move the parabola to the left as well as how to move it to the right.

g) Find a way to change the equation to make the $y = x^2$ parabola *move 3 units to the left* and *stretch vertically*, as in part (c).Your new parabola might look like $y = 4x^2$, except that the vertex should be at the point (–3, 0). Record the equations you try, along with their results. Comment about how to move the parabola to the left as well as how to move it to the right.

h) Finally, find a way to change the equation to make the $y = x^2$ parabola *vertically compressed*, *open down*, *move 6 units up*, and *move two units to the left*. Where is the vertex of your new parabola? Record the equations you try, their results, and your comments on how each part of the equation affects its graph.

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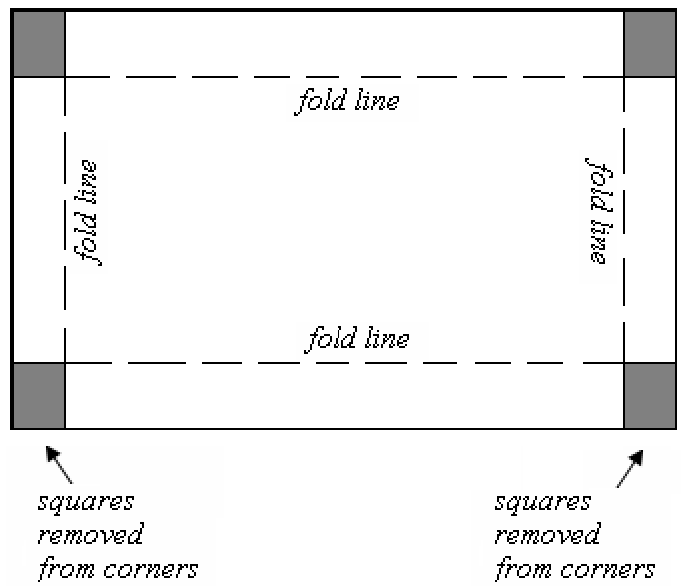
HENLEY'S CHOCOLATES

Henley Chocolates is famous for its mini chocolate truffles, which are packaged in foil covered boxes. The base of each box is created by cutting squares that are 4 centimeters on an edge from each corner of a rectangular piece of cardboard and folding the cardboard edges up to create a rectangular prism 4 centimeters deep. A matching lid is constructed in a similar manner, but, for this task, we focus on the base, which is illustrated in the diagrams below.

For the base of the truffle box, paper tape is used to join the cut edges at each corner. Then the inside and outside of the truffle box base are covered in foil.

Henley Chocolates sells to a variety retailers and creates specific box sizes in response to requests from particular clients. However, Henley Chocolates requires that their truffle boxes always be 4 cm deep and that, in order to preserve the distinctive shape associated with Henley Chocolates, the bottom of each truffle box be a rectangle that is two and one-half times as long as it is wide.



1. Henley Chocolates restricts box sizes to those which will hold plastic trays for a whole number of mini truffles. A box needs to be at least 2 centimeters wide to hold one row of mini truffles. Let L denote the length of a piece of cardboard from which a truffle box is made. What value of L corresponds to a finished box base for which the bottom is a rectangle that is 2 centimeters wide?

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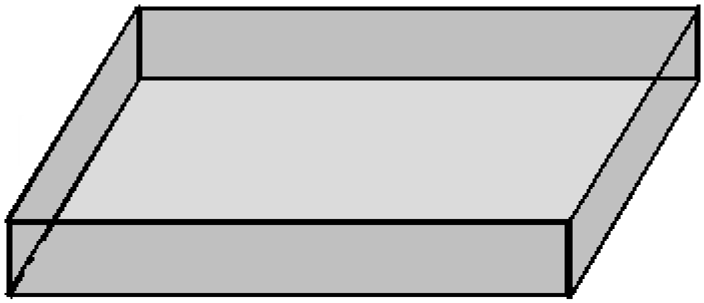

base of the truffle box

2. Henley Chocolates has a maximum size box of mini truffles that it will produce for retail sale. For this box, the bottom of the truffle box base is a rectangle that is 50 centimeters long. What are the dimensions of the piece of cardboard from which this size truffle box base is made?

3. Since all the mini truffle boxes are 4 centimeters deep, each box holds two layers of mini truffles. Thus, the number of truffles that can be packaged in a box depends the number of truffles that can be in one layer, and, hence, on the area of the bottom of the box. Let $A(x)$ denote the area, in square centimeters, of the rectangular bottom of a truffle box base. Write a formula for $A(x)$ in terms of the length L, in centimeters, of the piece of cardboard from which the truffle box base is constructed.

4. Although Henley Chocolates restricts truffle box sizes to those that fit the plastic trays for a whole number of mini truffles, the engineers responsible for box design find it simpler to study the function A on the domain of all real number values of L in the interval from the minimum value of L found in item 1 to the maximum value of L found in item 2. State this interval of L values as studied by the engineers at Henley Chocolates.

The next few items depart from Henley Chocolates to explore graph transformations that will give us insight about the function A for the area of the bottom of a mini truffle box. We will return to the function A in item 9.

5. **Use technology to graph** each of the following functions on the same axes with the graph of $f(x) = x^2$. Use a new set of axes for each function listed below, but repeat the graph of f each time. For each function listed, describe a rigid transformation of the graph of f that results in the graph of the given function. Make a conjecture about the graph of $y = (x-h)^2$, where h is any real number.

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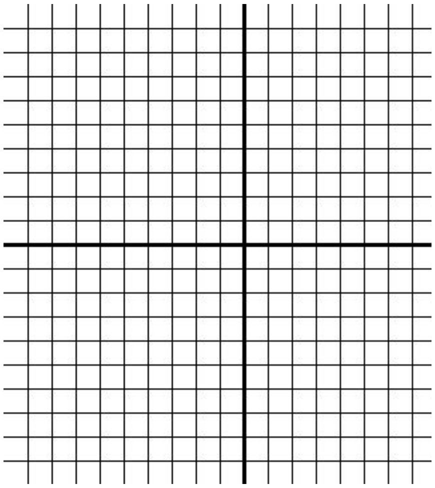
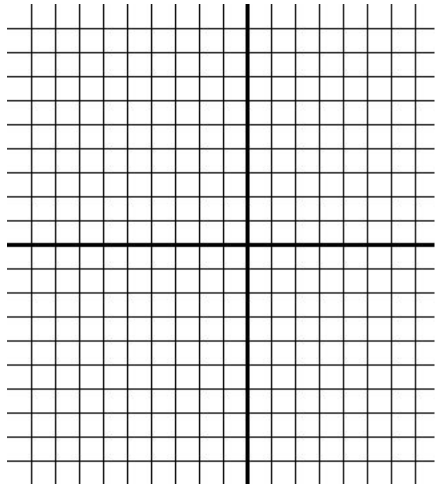
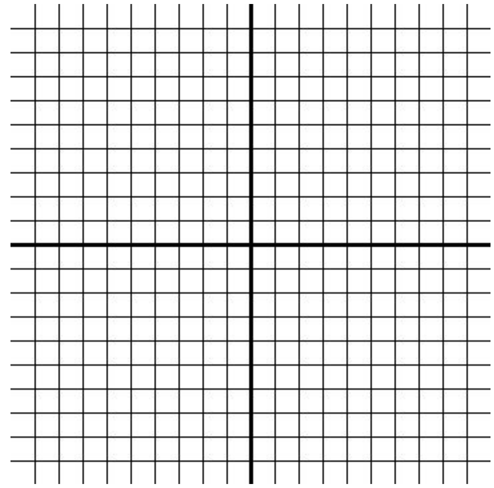
a. $y = (x-3)^2$



b. $y = (x-6)^2$

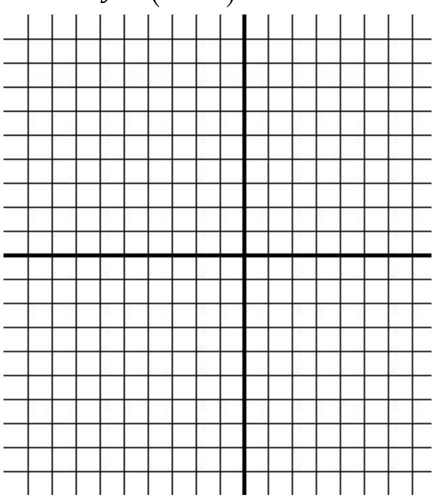


c. $y = (x-8)^2$

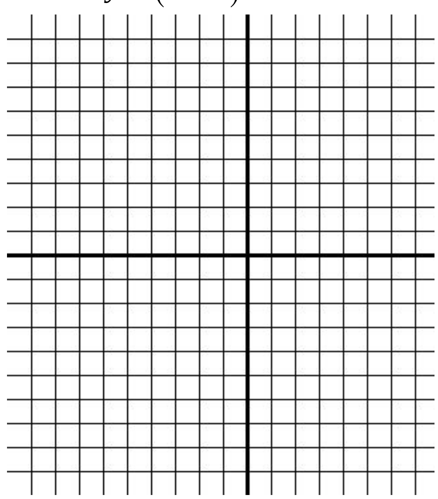


6. **Use technology to graph** each of the following functions on the same axes with the graph of $f(x) = x^2$. Use a new set of axes for each function listed below, but repeat the graph of f each time. For each function listed, describe a rigid transformation of the graph of f that results in the graph of the given function.

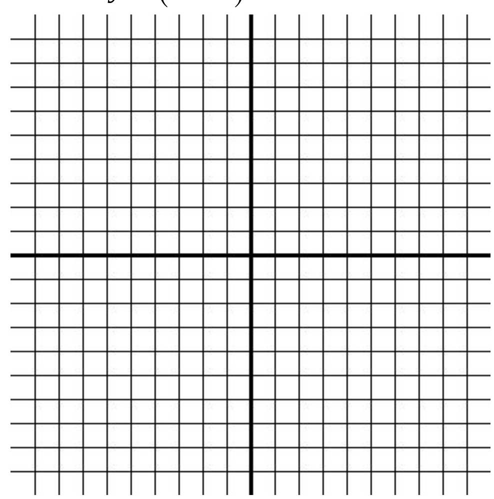
a. $y = (x+2)^2$



b. $y = (x+5)^2$



c. $y = (x+9)^2$



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7. We can view the exercises in item 5 as taking a function, in this case the function, $f(x) = x^2$, and replacing the "x" in the formula with "$x - h$". We can view the exercises in item 6 as replacing the "x" in the formula with "$x + h$", but we can also view these exercises as replacing the "x" in the formula with "$x - h$".
   a. How can this be done?

   b. Does your conjecture from item 5 agree with the transformations you described for item 6? If so, explain how it works. If not, adjust the statement of your conjecture to include these examples also.

   c. What do you think the will happen if we replace the "x" in the formula with "$x - h$" for other functions in our basic family of functions? Have you seen any examples of such replacements before?

8. For each pair of functions below, predict how you think the graphs will be related and then **use technology to graph** the two functions on the same axes and check your prediction.

Functions to Graph	Predictions	True/False
a. $y = x^2$ and $y = 3x^2$		
b. $y = 3x^2$ and $y = 3(x-4)^2$		
c. $y = \frac{1}{2}x^2$ and $y = -\frac{1}{2}x^2 + 5$		
d. $y = -0.75x^2$ and $y = 0.75(x+6)^2$		
e. $y = 2x^2$ and $y = -2(x-5)^2 + 7$		

Now we return to the function studied by the engineers at Henley Chocolates.

9. Let g be the function with the same formula as the formula for function A but with domain all real numbers. Describe the transformations of the function f, the square function, that will produce the graph of the function g. Use **technology to graph** f and g on the same axes to check that the graphs match your description of the described transformations.

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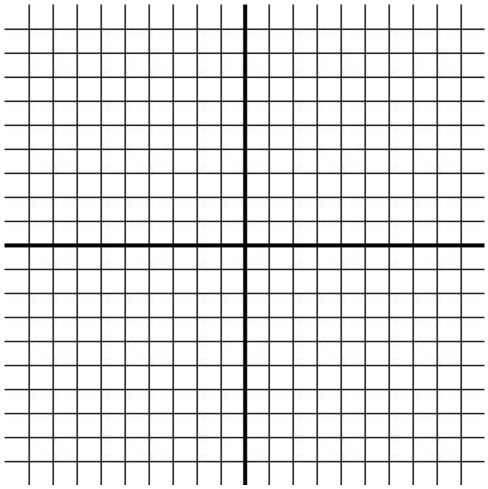
10. Describe the graph of the function *A* in words and make a hand drawn sketch. Remember that you found the domain of the function in item 4. What is the range of the function *A*?



11. The engineers at Henley Chocolates responsible for box design have decided on two new box sizes that they will introduce for the next winter holiday season.
    a. The area of the bottom of the larger of the new boxes will be 640 square centimeters. Use the function *A* to write and solve an equation to find the length *L* of the cardboard need to make this new box.

    b. The area of the bottom of the smaller of the new boxes will be 40 square centimeters. Use the function *A* to write and solve an equation to find the length *L* of the cardboard need to make this new box.

12. How many mini-truffles do you think the engineers plan to put in each of the new boxes?

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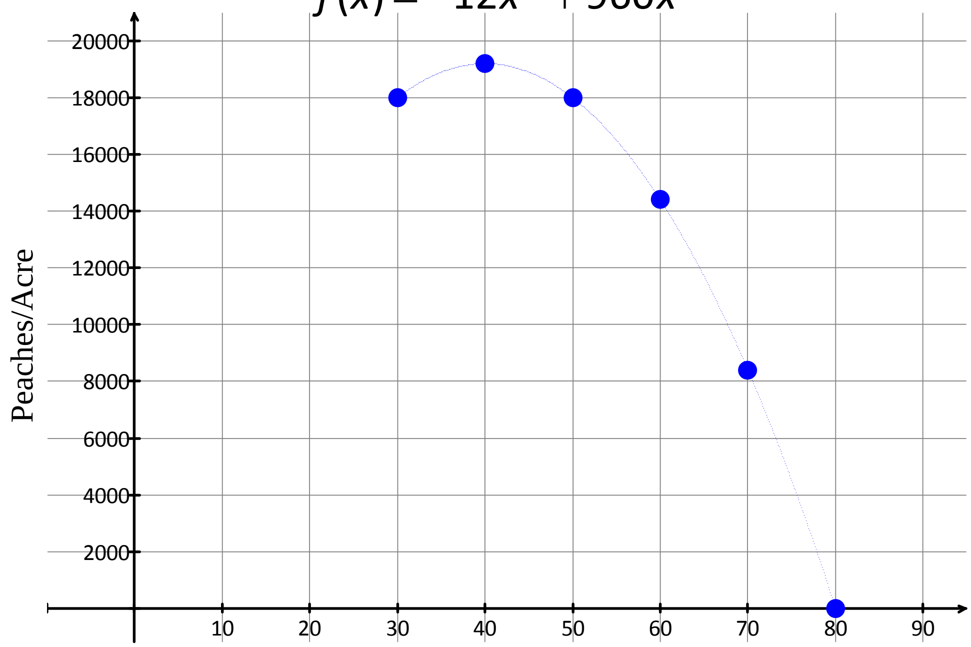
PAULA'S PEACHES: THE SEQUEL

In this task, we revisit Paula, the peach grower who wanted to expand her peach orchard. In the established part of her orchard, there are 30 trees per acre with an average yield of 600 peaches per tree. We saw that the number of peaches per acre could be modeled with a quadratic function. The function *f(x)* where *x* represents the number of trees and *f(x)* represents the number of peaches per acre. This is given below.

$$f(x) = -12x^2 + 960x$$



1. Remember that Paula wants to average more peaches. Her current yield with 30 trees is 18,000 peaches per acre and is represented by *f(30) = 18*,000. (Do you see this point on the graph?)

    a. Use the function above to write an **inequality** to express the average yield of peaches per acre to be at least 18,000.

    b. Since Paula desires at least 18,000 peaches per acre, draw a horizontal line showing her goal. Does this **line** represent her goal of **at least** 18,000 peaches per acre? Why or why not?

    c. Shade the region that represents her goal of **at least** 18,000 peaches per acre.

    d. Use the graph above to answer the following question: How many trees can Paula plant in order to yield at least 18,000 peaches? Write your solution as a compound inequality.

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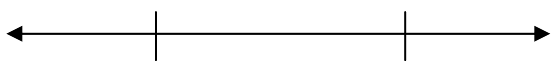
e. What is the domain of *f(x)*? In the context of Paula's Peaches, is your answer representative of the domain of *f(x)*?

2. Now let's find the answer to question 1 algebraically as opposed to graphically.
   a. Rewrite your inequality from 1a as an equation by replacing the inequality with an equal sign.

   b. The equation you just wrote is known as a **corresponding equation.** Solve the corresponding equation.

   c. When solving an inequality the solutions of the corresponding equation are called **critical values**. These values determine where the inequality is true or false. Place the critical values on the number line below. From your original inequalities, use an open circle if the value is not included and closed circle if value is included.



   d. You can now test your inequality by substituting any value from a given interval of the number line into the inequality created in 1a (original inequality). All intervals that test true are your solutions. Test each interval (left, middle, right) from your number line above. Then indicate your testing results by shading the appropriate intervals. Write your solution as a compound inequality.

   a. Compare your test in 2d to your answer in 1d. What do you notice?

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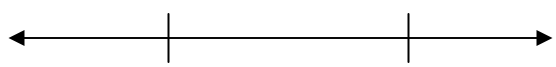
3. Paula must abide by a government regulation that states any orchard that produces more than 18,432 peaches will be taxed.

    a. Write an inequality to express when she will not be taxed.

    b. Write a *corresponding equation* and solve, finding the critical values.

    c. Now use a number line to solve the inequality as in part 2d. Write your answer as an inequality.



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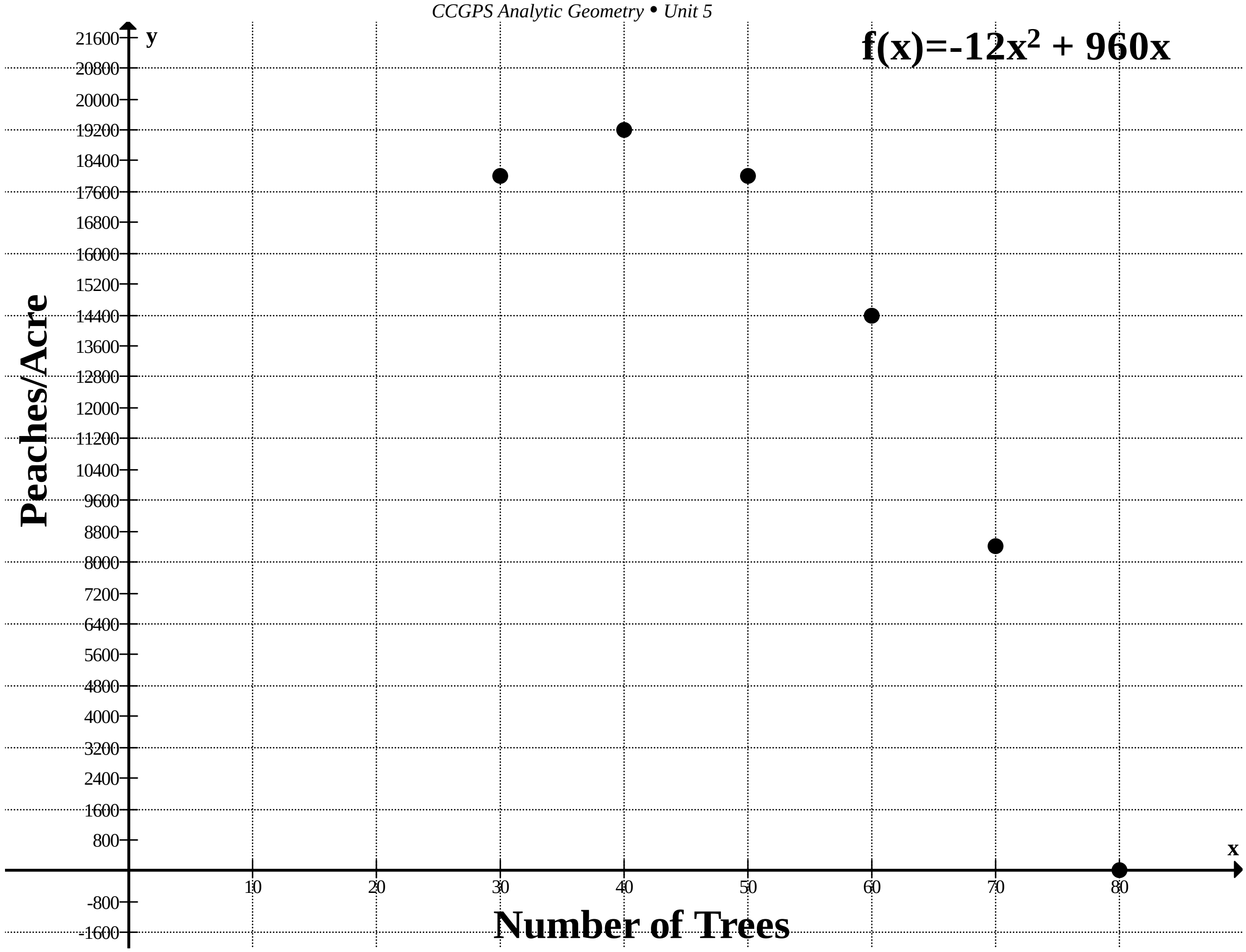

4. One year, a frost stunted production and the maximum possible yield was 14,400 peaches per acre.

   a. Write an inequality for this level of peach production using the function above.

   b. Since parabolas are symmetric, plot the reflective points on the graph above.

   c. Draw a horizontal line representing the maximum possible yield 14,400.

   d. Shade the region that represents her maximum yield of 14,400 peaches per acre.

   e. Are any of these values not in the original domain? Explain your answer and write your final solution as an inequality.

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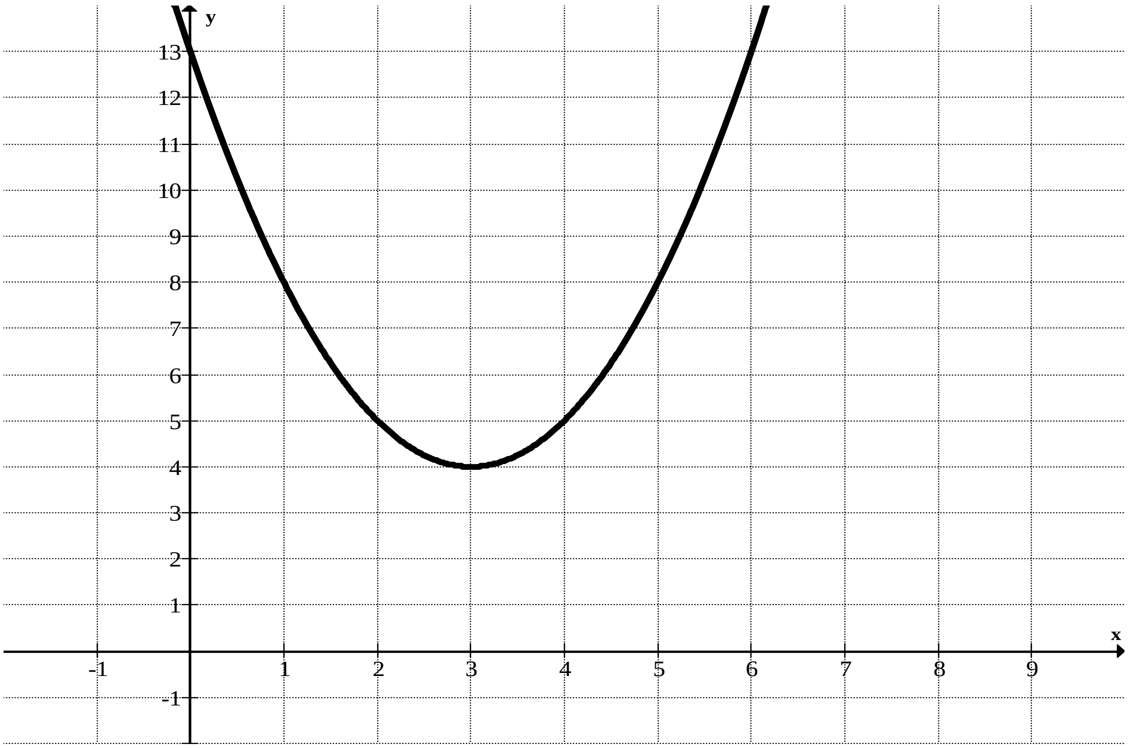
Practice Problems

For each graph below, solve the given inequality, writing your solution as an inequality. Each quadratic equation is given as $f(x)$.

1. $f(x) \geq 8$



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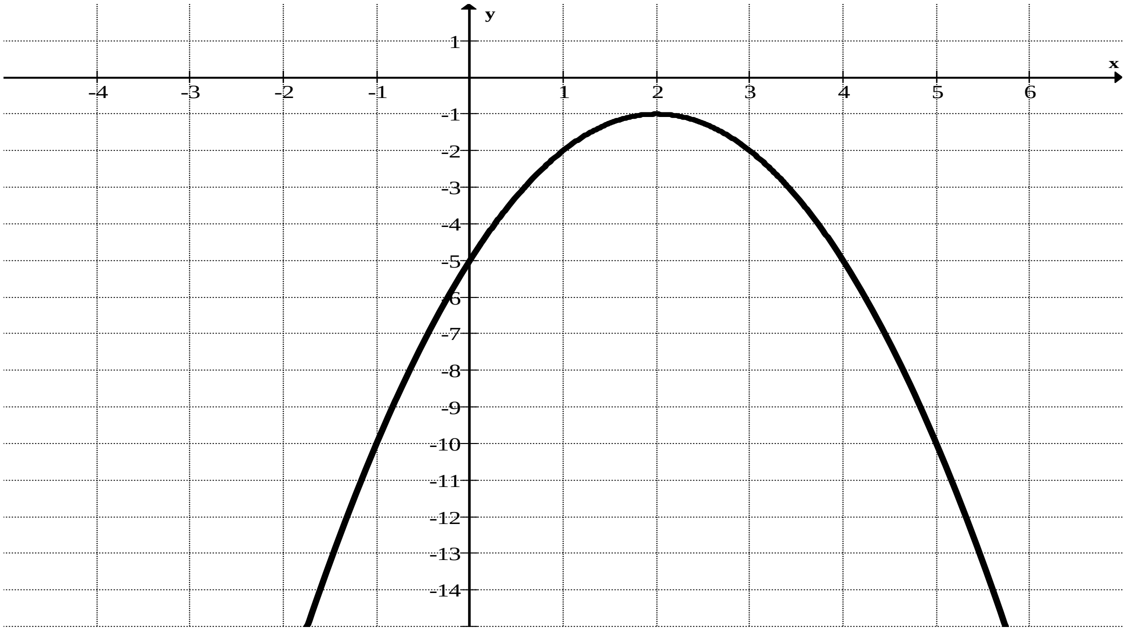
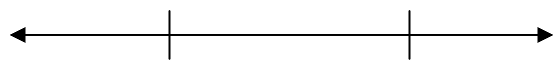
2. $f(x) \geq -10$

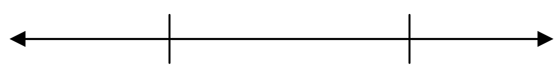


3. Solve the following inequalities algebraically.

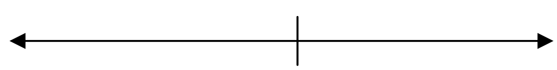
a.) $x^2 - 4x - 2 < -5$



b.) $3x^2 - 5x - 8 > 4$



c) $x^2 + 6x \geq -9$



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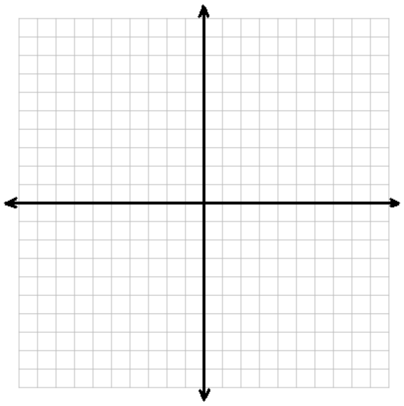
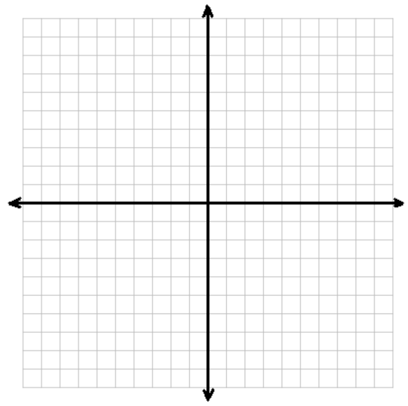
Parent Graphs Revisited

Complete a table, graph, and investigate the following functions.

a) $y = x^2 + 16x + 28$



b) $y = x^2 - 11x + 10$



State the following…

Domain:
Range:
Zeros:
Y-Intercept:
Interval of Increase:
Interval of Decrease:
Maximum:
Minimum:
End Behavior:
Even/Odd/Neither:

Domain:
Range:
Zeros:
Y-Intercept:
Interval of Increase:
Interval of Decrease:
Maximum:
Minimum:
End Behavior:
Even/Odd/Neither:

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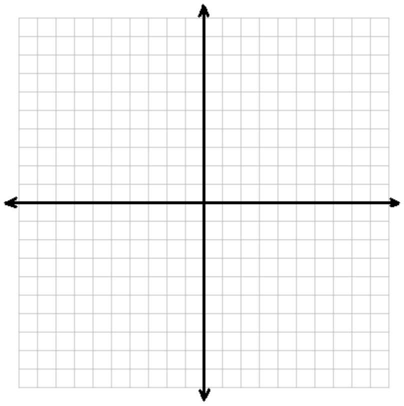
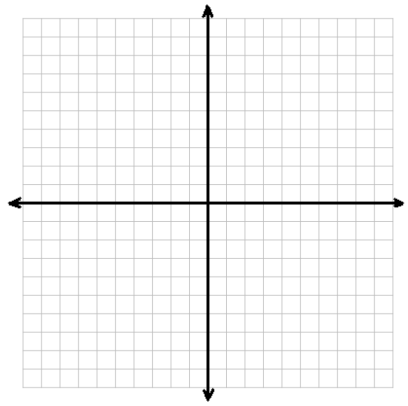
Complete a table, graph, and investigate the following functions.

c) $y = x^2 - 5x + 6$



d) $y = 5x^2 - 10x + 20$



State the following…

Domain:
Range:
Zeros:
Y-Intercept:
Interval of Increase:
Interval of Decrease:
Maximum:
Minimum:
End Behavior:
Even/Odd/Neither:

Domain:
Range:
Zeros:
Y-Intercept:
Interval of Increase:
Interval of Decrease:
Maximum:
Minimum:
End Behavior:
Even/Odd/Neither:

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